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Phase structure of non-commutative field theories and spinning brane bound states

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Abstract: General spinning brane bound states are constructed, along with their near-horizon limits which are relevant as dual descriptions of non-commutative field theories. For the spinning D-brane world volume theories with a B-field a general analysis of the gauge coupling phase structure is given, exhibiting various novel features, already at the level of zero angular momenta. We show that the thermodynamics is equivalent to the commutative case at large N and we discuss the possibility and consequences of finite N. As an application of the general analysis, the range of validity of the thermodynamics for the NCSYM is discussed. In view of the recently conjectured existence of a 7-dimensional NCSYM, the thermodynamics of the spinning D6-brane theory, for which a stable region can be found, is presented in detail. Corresponding results for the spinning M5-M2 brane bound state, including the near-horizon limit and thermodynamics, are given as well.

Keywords: Duality in Gauge Field Theories, p-branes, D-branes, Black Holes in String Theory.

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Contents

1. Introduction

Non-commutative geometry appears naturally in certain limits of string theory with a background NSNS B-field, as first discovered in the context of M(atrix) theory [1]. Recently, it has been shown [2] that non-commutative super Yang-Mills (NCSYM) theory directly appears from open string interactions, as suggested in earlier studies [3] of the subject. More specifically, NCSYM appears [2] in a special low-energy limit of the world-volume theory of N coinciding Dp-branes in the presence of a NSNS B-field. This fact has been used to extend the correspondence between near-horizon Dp-brane supergravity solutions and super Yang-Mills (SYM) theories in $p+1$ dimensions [4, 5, 6], to a correspondence between near-horizon Dp-brane supergravity solutions with a non-zero NSNS B-field and NCSYM in $p+1$ dimensions [7]–[10]. See also refs. [11]–[15] for further recent and related studies of non-commutative geometry in string theory.

The purpose of this paper is two-fold:

(i) Extend the analysis of the NCSYM phase structure given in ref. [10], to a more general path in the gauge theory phase space and use this to study the validity of D-brane thermodynamics for the NCSYM.

(ii) Construct spinning D-brane bound state solutions and use their near-horizon limit to analyze the thermodynamics of NCSYM, extending our recent work [16]. Modification of the gauge coupling phase structure of (i) due to the rotation will be considered as well. Since angular momenta and velocities on the supergravity side correspond to R-charges and R-voltages on the NC-SYM side, the thermodynamics of these spinning brane solutions with a B-field may provide further insights into NCSYM.

We start in section 2 with constructing general spinning D-brane bound state solutions, by applying a set of T-dualities to the general spinning Dp-brane solutions [17, 16]. The resulting backgrounds[1] are bound states of spinning $D(p-2k)$-branes, $k=0,\ldots,m$, with $2m$ the rank of the NSNS B-field. For a given p, the solution is spinning in the $9-p$ dimensional transverse space, and we show that, except for additional charges and chemical potentials of the lower branes in the bound state, the thermodynamics is equivalent to that of a spinning Dp-brane.

In section 3 we construct the near-horizon limit of the general spinning D-brane bound state, and discuss the conditions in order for the near-horizon solution to describe the dual NCSYM, focusing for simplicity on the non-rotating case first. The phase space of the NCSYM at fixed N is parametrized by the YM coupling constant $g_{\rm YM}$, the gauge theory energy scale r and the non-commutativity parameters b_k, $k=1\ldots m$, which enter the position commutators [7, 10]

$$[y^{2k-1}, y^{2k}] = i b_k \,, \qquad k=1,\ldots,m\,. \tag{1.1}$$

By considering a general type of path in phase space, we discuss certain general features of the resulting phase diagrams in terms of the effective coupling $g_{\rm eff}$. This analysis depends on the dimension p of the Dp-brane and on the rank $2m$ of the NSNS B-field. We find four types of phase diagrams, and analyze in detail which phase diagram is relevant for the chosen path and region of phase space. For each value of p and $m>0$, we establish that a path and region of phase space can be chosen such that the phase structure of any of the four phase diagrams can be realized. We also

[1] For zero angular momentum these spinning bound state solutions reduce to the bound states given in refs. [18, 19].

determine under which conditions the description with finite N is valid. Our analysis includes the case considered in [10] and, as another case of special interest, the path in phase space along which the intensive thermodynamic quantities are invariant, which is used to examine the validity of the thermodynamics. We end section 3 by discussing the effects of angular momenta on the gauge coupling phase structure.

In section 4 we discuss the thermodynamics of the near-horizon solutions and their corresponding dual NCSYM theories. As was argued for the non-rotating case [7, 10, 12, 13], it is seen that the thermodynamic quantities are the same as for commutative SYM case, with the coupling constant $g^2_{\rm YM}$ of the latter replaced by $g^2_{\rm YM} \prod_{k=1}^m b_k$ in the NCSYM case. Then, using the results of section 3, the range of validity of the thermodynamics is discussed, explaining specifically the cases for which a) the coupling can go all the way to infinity and b) finite N is allowed. Moreover, we find the region of thermodynamic phase space where both these properties can hold. We conclude this section with a detailed analysis of the thermodynamics of the spinning D6-brane theory, adding to the analysis of the near-horizon D6-brane in ref. [16]. This is of interest in view of the recent discovery that the D6-brane theory decouples from gravity for $m \geq 1$ [7, 10], and since there is a critical angular momentum density above which the spinning D6-brane is stable in the canonical ensemble [16].

For completeness, section 5 gives the corresponding results for the spinning M5-M2 brane bound state, which can be obtained by lifting the spinning D4-D2 brane bound state to M-theory. The asymptotically-flat solution is given with its thermodynamics as well as the near-horizon solution, which is dual to the non-commutative (2,0) theory. As expected the thermodynamics is again independent of the non-commutativity parameter. Finally, section 6 presents conclusions and discussion. Appendix A reviews the T-duality transformations used to obtain our spinning bound state solutions and appendix B provides some details on the RR gauge potentials of these solutions and their near-horizon limit.

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2. Spinning D-brane bound states

The spinning D-brane bound states that we will present are solutions of the low-energy effective action of type II string theory in the string frame

$$I = \frac{1}{16\pi G} \int d^{10}x \sqrt{g} \left[e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_3^2 \right) - \sum_p \frac{1}{2(p+2)!} F_{p+2}^2 \right] + I_{WZ} \,. \tag{2.1}$$

Here $H_3 = dB_2$ and $F_{p+2} = dA_{p+1} + \cdots$ are the field strengths of the NSNS 2-form and $(p+1)$-form RR gauge potentials respectively, where the dots denote terms involving B_2. The integers p are even (odd) for type IIA (IIB) and I_{WZ} denotes topological terms involving B_2 and A_{p+1}. Using T-duality as a solution generating technique (see appendix A) one finds from the general spinning D-branes [17, 16] the corresponding spinning D-brane bound states with a non-zero B-field. In the notation of [16] these new solutions take the following form: the metric is

$$\begin{aligned} ds^2 &= H^{-1/2}\left(-f\,dt^2+\sum_{k=1}^{m}D_k\Big[(dy^{2k-1})^2+(dy^{2k})^2\Big]+\sum_{i=2m+1}^{p}(dy^i)^2\right)+\\ &\quad + H^{1/2}\left(\bar{f}^{-1}K_{9-p}\,dr^2+\Lambda_{\alpha\beta}d\eta^\alpha d\eta^\beta\right)+\\ &\quad + H^{-1/2}\frac{1}{W_p}\frac{r_0^{7-p}}{r^{7-p}}\left(\sum_{i,j=1}^{n}l_i l_j\mu_i^2\mu_j^2 d\phi_i d\phi_j-2\cosh\alpha\sum_{i=1}^{n}l_i\mu_i^2 dt d\phi_i\right), \end{aligned} \tag{2.2}$$

while the dilaton takes the form

$$e^{2\phi}=H^{(3-p)/2}\prod_{k=1}^{m}D_k\,. \tag{2.3}$$

The NSNS B-field has rank $2m\le p$ and is given by

$$B_{2k-1,2k}=\tan\theta_k\Big(H^{-1}D_k-1\Big)\,,\qquad k=1,\ldots,m \tag{2.4}$$

and we refer to appendix B for the form of the non-zero RR gauge potentials A_{p-2k+1}, $k=0,\ldots,m$.

The functions[2] entering this background are given by

$$L_{9-p}=\prod_{i=1}^{n}\left(1+\frac{l_i^2}{r^2}\right)\,,\qquad\qquad H=1+\frac{1}{W_p}\frac{r_0^{7-p}\sinh^2\alpha}{r^{7-p}} \tag{2.5a}$$

$$f=1-\frac{1}{W_p}\frac{r_0^{7-p}}{r^{7-p}}\,,\qquad\qquad \bar{f}=1-\frac{1}{L_{9-p}}\frac{r_0^{7-p}}{r^{7-p}}\,, \tag{2.5b}$$

where

$$W_p=K_{9-p}L_{9-p}\,, \tag{2.6}$$

with L_{9-p} defined in (2.5a) and K_{9-p} entering the flat transverse space metric

$$\sum_{a=1}^{9-p}(dx^a)^2=K_{9-p}\,dr^2+\Lambda_{\alpha\beta}d\eta^\alpha d\eta^\beta \tag{2.7}$$

(See ref. [16, appendix B] for the explicit expressions of K_{p-9}, $\Lambda_{\alpha\beta}$ and the μ_i in (2.2)). We have also defined

$$D_k=\Big(\sin^2\theta_k H^{-1}+\cos^2\theta_k\Big)^{-1}\,,\qquad k=1,\ldots,m \tag{2.8}$$

[2] In ref. [16] the functions K_d, L_d are labeled by the transverse dimension d, which is equal to $9-p$ for Dp-branes, and we have kept the same definitions for these.

Finally we recall the definition

$$h^{7-p} = r_0^{7-p} \cosh\alpha \sinh\alpha \tag{2.9}$$

and the relations

$$16\pi G = (2\pi)^7 g_s^2 l_s^8 \,, \qquad h^{7-p} = \frac{(2\pi)^{7-p} N g_s l_s^{7-p}}{(7-p)V(S^{8-p})} \prod_{k=1}^{m} (\cos\theta_k)^{-1} \,, \tag{2.10}$$

where l_s is the string length, g_s the string coupling and $V(S^{8-p})$ the volume of the unit $(8-p)$-sphere. The second relation in (2.10) is a consequence of charge quantization of the Dp-brane, where N is the number of coincident Dp-branes.

These solutions represent spinning bound states of those branes that carry charges under the non-zero RR fields, i.e. of D$(p-2k)$, $k = 0, \ldots, m$. The background depends on the non-extremality parameter r_0, charge parameter α, the angular momenta l_i, $i = 1, \ldots, n$ ($n \equiv [(9-p)/2]$) and the angles θ_k, $k = 1, \ldots, m$. For zero angular momentum the bound state solutions of [18, 19] are recovered.

Besides the charges and chemical potentials, the thermodynamic quantities of the bound state solution are not affected by the non-zero B-field in that they are given by the corresponding expressions of the spinning Dp-brane [16]

$$M = \frac{V_p V(S^{8-p})}{16\pi G} r_0^{7-p} \Big(8 - p + (7-p)\sinh^2\alpha \Big) \tag{2.11a}$$

$$T = \frac{7-p-2\kappa}{4\pi r_H \cosh\alpha} \,, \qquad S = \frac{V_p V(S^{8-p})}{4G} r_0^{7-p} r_H \cosh\alpha \tag{2.11b}$$

$$\Omega_i = \frac{l_i}{(l_i^2 + r_H^2)\cosh\alpha} \,, \qquad J_i = \frac{V_p V(S^{8-p})}{8\pi G} r_0^{7-p} l_i \cosh\alpha \,. \tag{2.11c}$$

We refer to appendix B for the expressions of the charges and chemical potentials, which satisfy

$$\sum_{k=0}^{m} \sum_{\alpha} \mu_{p-2k}^{(\alpha)} Q_{p-2k}^{(\alpha)} = \mu Q \tag{2.12a}$$

$$\mu = \tanh\alpha \,, \qquad Q = \frac{V_p V(S^{8-p})}{16\pi G} r_0^{7-p} (7-p) \sinh\alpha \cosh\alpha \,, \tag{2.12b}$$

where for a given k, α labels the distinct D$(p-2k)$ branes embedded in the Dp-brane. In (2.11) the horizon radius r_H and the coefficient κ are given by

$$\prod_{i=1}^{n} \left(1 + \frac{l_i^2}{r_H^2} \right) r_H^{7-p} = r_0^{7-p} \,, \qquad \kappa = \sum_{i=1}^{n} \frac{l_i^2}{l_i^2 + r_H^2} \tag{2.13}$$

The first law of thermodynamics reads

$$dM = TdS + \sum_{i=1}^{n} \Omega_i dJ_i + \sum_{k=0}^{m} \sum_{\alpha} \mu_{p-2k}^{(\alpha)} dQ_{p-2k}^{(\alpha)} \,, \qquad M = M\left(S, \{J_i\}, \{Q_{p-2k}^{(\alpha)}\} \right) . \tag{2.14}$$

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It then follows from (2.11) and (2.12) that the integrated Smarr formula

$$(7-p)M = (8-p)TS + (7-p)\mu Q + (8-p)\sum_{i=1}^{n} \Omega_i J_i \tag{2.15}$$

is satisfied.

Note that from the D3-D1 or D5-D3-D1 bound state, obtained for $p = 3$ or 5, it is not difficult to obtain the corresponding background of D3-NS1 or NS5-D3-NS1 using type IIB S-duality. The thermodynamic quantities will then remain unchanged. For zero angular momentum these solutions can be found in [20] and [10].

3. The near-horizon limit

In this section we consider the near-horizon limit of the spinning D-brane bound state solutions found in section 2. In section 3.1 we obtain the near-horizon solution by taking the appropriate limit. In section 3.2 we review some properties of the dual non-commutative field theories corresponding to these near-horizon solutions, while section 3.3 gives a detailed analysis of the gauge coupling phase structure. Finally, section 3.4 describes the effect of non-zero angular momenta in the dual field theory, and discusses the induced modifications in the analysis of the gauge coupling phase structure, presented in section 3.3 for zero angular momenta.

3.1 Near-horizon solutions

We start by constructing the near-horizon limit of the spinning bound state (2.2)–(2.4) in which the magnetic field (2.4) is taken to infinity in such a manner that a finite rescaled value is obtained after taking the limit. As reviewed in section 3.2, this corresponds to a non-commutative field theory on the world-volume of the Dp-brane. For the non-rotating case, this limit was found in refs. [2, 7, 8, 10]. This limit crucially depends on the rank of the B-field, denoted by $2m \leq p$.

The near-horizon limit is defined by letting the string length $l_s \to 0$ accompanied by the rescalings

$$r = \frac{r_{\rm old}}{l_s^2}\,, \qquad r_0 = \frac{(r_0)_{\rm old}}{l_s^2}\,, \qquad l_i = \frac{(l_i)_{\rm old}}{l_s^2}\,, \qquad h^{7-p} = \frac{h_{\rm old}^{7-p}}{l_s^{10-2p}} \tag{3.1a}$$

$$ds^2 = \frac{(ds^2)_{\rm old}}{l_s^2}\,, \qquad e^{\phi} = l_s^{3-p+2m} e^{\phi_{\rm old}}\,, \qquad G = \frac{G_{\rm old}}{l_s^{14-2p+4m}} \tag{3.1b}$$

and the rescalings

$$b_k = l_s^2 \tan\theta_k\,, \qquad y^{2k-1} = \frac{b_k}{l_s^2}(y^{2k-1})_{\rm old}\,, \qquad y^{2k} = \frac{b_k}{l_s^2}(y^{2k})_{\rm old} \tag{3.2a}$$

$$B_{2k-1,2k} = l_s^{-2}(B_{2k-1,2k})_{\rm old}\,, \qquad k = 1,\ldots,m\,, \tag{3.2b}$$

where the quantities on the left-hand side in (3.1), (3.2) are kept fixed, and the quantities on the right-hand side (labelled with subscript "old" except for θ_k) are those that enter the asymptotically-flat solutions of section 2. The corresponding rescalings of the RR gauge potentials are given in (B.11), which together with (3.1b) and (3.2b) leave the low-energy effective action (2.1) invariant. Note that for $m = 0$ (i.e. zero B-field) the rescalings (3.1) correctly reduce to the rescalings of the spinning Dp-brane solutions described in [16].

Taking the near-horizon limit, the following near-horizon spinning Dp-brane solution with non-zero B-field is obtained

$$ds^2 = H^{-1/2}\left(-f\,dt^2 + \sum_{k=1}^{m} D_k\left[(dy^{2k-1})^2 + (dy^{2k})^2\right] + \sum_{i=2m+1}^{p}(dy^i)^2\right) + \\ + H^{1/2}\left(\bar{f}^{-1}K_{9-p}dr^2 + \Lambda_{\alpha\beta}d\eta^\alpha d\eta^\beta\right) - 2H^{-1/2}\frac{h^{\frac{7-p}{2}}r_0^{\frac{7-p}{2}}}{W_p r^{7-p}}\sum_{i=1}^{n} l_i\mu_i^2 dt d\phi_i \quad (3.3a)$$

$$e^{2\phi} = H^{(3-p)/2}\prod_{k=1}^{m} b_k^2 D_k \quad (3.3b)$$

$$B_{2k-1,2k} = \frac{1}{b_k}\frac{a_k^{7-p}W_p r^{7-p}}{1 + a_k^{7-p}W_p r^{7-p}}\,, \qquad k = 1, \ldots, m\,, \quad (3.3c)$$

where now

$$H = \frac{h^{7-p}}{W_p r^{7-p}}\,, \qquad D_k = \left(1 + a_k^{7-p}W_p r^{7-p}\right)^{-1}\,, \qquad k = 1, \ldots, m \quad (3.4)$$

and we have defined

$$a_k^{7-p} = \frac{b_k^2}{h^{7-p}}\,, \qquad k = 1, \ldots, m\,. \quad (3.5)$$

The functions K_{9-p}, W_p and $\Lambda_{\alpha\beta}$ are not affected by the rescaling and hence as in section 2. Note that a gauge transformation has been made that removes the constant part in B. The corresponding expression for the RR gauge potentials of the near-horizon solution can be obtained in principle with the data of appendix B. For zero angular momenta the background reduces to the near-horizon solutions of [7, 8, 10].

3.2 The dual field theories

We continue with describing the map between the variables of the near-horizon solution and the dual field theory variables. This is done for zero angular momenta ($l_i = 0$), but in section 3.4 the modifications arising from non-zero l_i will be discussed. We also comment on the validity of the thermodynamics of the dual field theories and review the conditions under which gravity decouples from the world-volume theory on the brane in the near-horizon limit.

The zero slope limit $l_s \to 0$ in the presence of a B-field with rank $2m > 0$, considered in section 3.1, gives at low energies a world-volume theory that is described by the Dp-brane Born-Infeld action with a non-zero B-field. The latter is equivalent [2] to a non-commutative supersymmetric Yang-Mills (NCSYM) theory in $p+1$ dimensions with 16 supercharges, where the non-commutativity of the coordinate pair (y^{2k-1}, y^{2k}) is given by $[y^{2k-1}, y^{2k}] = ib_k$ [2, 7, 10]. The NCSYM theory on the brane has a coupling constant $g_{\rm YM}$ given by

$$g_{\rm YM}^2 = (2\pi)^{p-2} g_s l_s^{p-3-2m} = (2\pi)^{p-2} \bar{g}_s \,, \qquad \bar{g}_s = g_s l_s^{p-3-2m} \,, \tag{3.6}$$

where $\bar{g}_s$ is the rescaled string coupling constant. From (2.10) and the rescalings (3.1) and (3.2) we find in the near-horizon limit the new quantities

$$h^{7-p} = \frac{(2\pi)^{9-2p} g_{\rm YM}^2 N \prod_{k=1}^m b_k}{(7-p) V(S^{8-p})} \,, \qquad 16\pi G = (2\pi)^7 \bar{g}_s^2 = (2\pi)^{11-2p} g_{\rm YM}^4 \,. \tag{3.7}$$

Following [5, 10] we introduce the effective gauge coupling of the world-volume theory

$$g_{\rm eff}^2 = g_{\rm YM}^2 N \left(\prod_{k=1}^m b_k \right) r^{p-3} \tag{3.8}$$

where the rescaled radius r has the interpretation of the effective energy scale of the field theory, being the expectation value of the Higgs field [4, 5]. Since the curvature of the metric (3.3a) is of order $1/g_{\rm eff}$, the requirement that curvatures be small imposes the restriction $g_{\rm eff} \gg 1$, so that one needs to be in the strong coupling region for the supergravity description of the D-brane world-volume theory to hold. The perturbative description of the world-volume theory on the D-brane is instead valid at weak coupling $g_{\rm eff} \ll 1$.

We also introduce the effective non-commutativity (NC) parameters [10]

$$a_k^{\rm eff} = a_k r \,, \qquad k = 1, \ldots, m \,. \tag{3.9}$$

Then, the region $a_k^{\rm eff} \ll 1$ corresponds to the coordinate pair (y^{2k-1}, y^{2k}) being commutative, while in the opposite region $a_k^{\rm eff} \gg 1$ these coordinates are non-commutative. We finally introduce the effective string coupling [5, 10]

$$g_s^{\rm eff} = \bar{g}_s e^{\phi} \tag{3.10}$$

and from (3.3a) and (3.6) it follows that

$$g_s^{\rm eff} \ll 1 \;\Leftrightarrow\; g_{\rm eff}^2 \ll N^{\frac{4}{7-p}} \prod_{k=1}^m \left(1 + (a_k^{\rm eff})^{7-p} \right)^{\frac{2}{7-p}} \,. \tag{3.11}$$

The fact that the NC parameters $a_k^{\rm eff}$ enter in the effective string coupling [10] has interesting consequences, as we will see shortly.

The requirements $g_{\text{eff}} \gg 1$ and (3.11) in principle determine for which values of N, g_{YM}, r and b_k, $k = 1, \ldots, m$, the dual field theory is described by the D-brane in the near-horizon limit. For the sequel, it is important to note that the D-brane thermodynamics in the near-horizon limit gives the thermodynamics of the dual field theory precisely when we set $r = r_0$ in the relevant expressions [5]. In the special case of the D3 and D5-brane, we note that the condition $g_s^{\text{eff}} \ll 1$ (with $r = r_0$) is no longer relevant for the thermodynamics, since their S-dual branes, being the D3 and NS5-brane, have the same thermodynamics.[3] As a consequence, for the D3 and D5-brane case there is no bound on g_s^{eff} as far as the thermodynamics is concerned.

It is also important to note that for the Dp-branes with $p \leq 5$ gravity decouples from the D-brane world-volume theory for all values of m, while for the D6-brane this only happens when $m \geq 1$ [7, 10]. In other words, for the D6-brane the presence of a non-zero B-field is crucial for the decoupling of gravity. Only when gravity decouples can we expect to have a well-defined field theory that is described by supergravity when the curvatures are small. As noted in [7, 10] the decoupling of gravity for the D6-brane with $m \geq 1$ implies the existence of a 7 dimensional non-commutative field theory. For this reason we will, after having discussed the thermodynamics and phase structure in generality, give a more detailed account in section 4.3 of what we can infer for this D6-brane theory.

3.3 Gauge coupling phase structure

We now proceed to find the gauge coupling phase structure of the D-brane world-volume theories in a more general setting than previously done. In [5] the gauge coupling phase structure was considered for the D-brane world-volume theories without B-field, that is, with $m = 0$. As we shall see, for $m > 0$ a much richer structure is observed. The phase diagrams obtained in [10] for $m > 0$ for the D2, D4, D5 and D6-brane will arise as a special case of the analysis we present here. In the following we take $2 \leq p \leq 6$ and $1 \leq m \leq [\frac{p+1}{2}]$.[4] We also set the angular momenta to zero, commenting in section 3.4 on the modifications due to non-zero l_i.

For simplicity, we set $b_k = b$, $k = 1, \ldots, m$, so that the phase space is parameterized by N, g_{YM}, r and b. Our aim is to study the phase structure in terms of the effective gauge coupling

$$z \equiv g_{\text{eff}}^2 \tag{3.12}$$

going from zero to infinity. However, there is actually a large freedom in the choice of path in phase space that one may follow, which we parametrize by

$$g_{\text{YM}}^2 \propto z^\alpha\,, \qquad r \propto z^\beta\,, \qquad b \propto z^\gamma \tag{3.13}$$

[3] This is explained for the spinning D5 and NS5-brane in ref. [16].

[4] Note that the analysis of this section also holds for $m = \frac{p+1}{2}$ if p is odd, when the euclidean background is considered as done in [10].

keeping N fixed. From (3.8) and the definition (3.12) it then follows that the scaling exponents α, β, γ obey the constraint

$$\alpha + (p-3)\beta + m\gamma = 1\,. \tag{3.14}$$

It also follows from (3.5) and (3.7)–(3.9) that as a function of z we can write

$$(a^{\mathrm{eff}})^{7-p} = \left(\frac{z}{z_{\mathrm{nc}}}\right)^{\eta}\,, \tag{3.15}$$

where

$$\eta = 4\beta + 2\gamma - 1 \tag{3.16}$$

and z_{nc} is a constant determined by (3.15). Since a^{eff} determines the (non)-commutativity of the theory, we find that for $\eta > 0$ the field theory is non-commutative for $z \gg z_{\mathrm{nc}}$ and commutative for $z \ll z_{\mathrm{nc}}$, while in the case $\eta < 0$ the field theory is commutative for $z \gg z_{\mathrm{nc}}$ and non-commutative for $z \ll z_{\mathrm{nc}}$.

Using (3.15) the condition (3.11) can be rewritten as

$$g_s^{\mathrm{eff}} \ll 1 \;\Leftrightarrow\; z \ll N^{\frac{4}{7-p}} \left(1 + \left(\frac{z}{z_{\mathrm{nc}}}\right)^{\eta}\right)^{\frac{2m}{7-p}} \tag{3.17}$$

which will enable us to analyze the phase structure in terms of z. One of the common features of this phase structure is that for $z \ll 1$ we have a perturbative description of the world-volume field theory, while for $z \gg 1$ the curvatures are small and the supergravity description of the world-volume theory is valid. If there is a transition point z_{t} where $g_s^{\mathrm{eff}} = 1$ then the supergravity Dp-brane description is valid in the range $1 \ll z \ll z_{\mathrm{t}}$ and for $z \gg z_{\mathrm{t}}$ we go to an S-dual brane description. In type IIB string theory this means the NS5-brane for the D5-brane, or the D3-brane itself in the case of the D3-brane. For type IIA string theory this means a supergravity solution up-lifted to 11-dimensional supergravity.[5] As described in [5] the world-volume field theories on the D-branes with $m = 0$ all have the phase structure with one transition point $z_{\mathrm{t}} \sim N^{\frac{4}{7-p}}$ where $g_s^{\mathrm{eff}} = 1$. The phase diagram corresponding to this phase structure is depicted in figure 1.

We divide the description of the gauge coupling phase structure into five cases, depending on the parameter η. Except for $\eta = 0$, each case is again subdivided into the two cases $z_{\mathrm{nc}} \ll N^{\frac{4}{7-p}}$ and $z_{\mathrm{nc}} \gg N^{\frac{4}{7-p}}$:

1. $\eta < 0$: here $z \ll z_{\mathrm{nc}}$ is the non-commutative sector, so that $z \ll 1$ gives a perturbative NCSYM description (assuming that $z_{\mathrm{nc}} > 1$).

[5] For the D2-brane we have an additional phase transition point in the 11-dimensional sector, namely the point where the supergravity solution becomes a localized M2-brane in 11 dimensions [5].

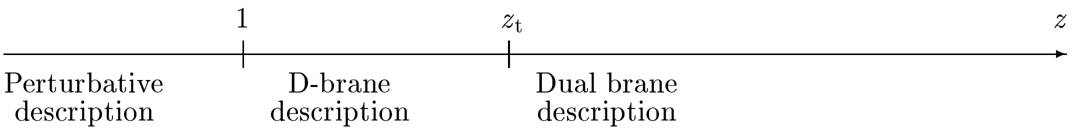

Figure 1: Phase diagram with one transition point z_t where $g_s^{\rm eff} = 1$.

Consider the case $z_{\rm nc} \ll N^{\frac{4}{7-p}}$. This means that the point $z = N^{\frac{4}{7-p}}$ lies in the commutative sector, so the supergravity Dp-brane description is valid for $1 \ll z \ll N^{\frac{4}{7-p}}$, and hence we need $N \gg 1$. For $z \gg N^{\frac{4}{7-p}}$ we go to the S-dual brane theory and since we are in the commutative sector the phase structure is the same as found in [5] and the phase diagram is depicted in figure 1 with $z_t \sim N^{\frac{4}{7-p}}$.

Consider the case $N^{\frac{4}{7-p}} \ll z_{\rm nc}$. Here the transition point z_t where $g_s^{\rm eff} = 1$ is between $N^{\frac{4}{7-p}}$ and $z_{\rm nc}$. Except for the shift in z_t the phase structure is the same as for $z_{\rm nc} \ll N^{\frac{4}{7-p}}$, thus the phase diagram is again figure 1. Remarkably, we can have N finite in this case.

2. $\eta = 0$: in this case $a^{\rm eff}$ in (3.15) is constant. There is a transition point $z_t \sim N^{\frac{4}{7-p}}(1 + (a^{\rm eff})^{7-p})^{\frac{2m}{7-p}}$ for which $g_s^{\rm eff} = 1$. Apart from a shift in z_t the phase structure is qualitatively the same as for $m = 0$, and thus as depicted in figure 1. This case is interesting since we can choose $a^{\rm eff} \gg 1$ so that the theory is non-commutative for all z. We can clearly have N finite if we choose $a^{\rm eff} \gg 1$.

3. $0 < \eta < \frac{7-p}{2m}$: here $z \gg z_{\rm nc}$ is the non-commutative sector, so that $z \ll 1$ gives a perturbative commutative SYM description (assuming that $z_{\rm nc} > 1$).

   Consider the case $N^{\frac{4}{7-p}} \ll z_{\rm nc}$. For this case the point $N^{\frac{4}{7-p}}$ lies in the commutative sector so the supergravity Dp-brane description is valid for $1 \ll z \ll N^{\frac{4}{7-p}}$, so that we need to require $N \gg 1$. Thus, apart from the non-commutativity the phase structure is the same as for $m = 0$, depicted in figure 1 with $z_t \sim N^{\frac{4}{7-p}}$.

   Consider the case $z_{\rm nc} \ll N^{\frac{4}{7-p}}$. Here a transition point z_t is found where $g_s^{\rm eff} = 1$, and z_t is either of the same order or larger than $N^{\frac{4}{7-p}}$. Again the phase structure is qualitatively the same as for $m = 0$, depicted in figure 1. If $z_{\rm nc} \ll 1$ it is possible to have N finite (for small η it should be $z_{\rm nc}^{\eta} \ll 1$).

4. $\eta = \frac{7-p}{2m}$: here $z \gg z_{\rm nc}$ is the non-commutative sector, so for $z \ll 1$ we have a perturbative commutative SYM description (assuming that $z_{\rm nc} > 1$).

   Consider the case $N^{\frac{4}{7-p}} \ll z_{\rm nc}$. The supergravity D$p$-brane description is valid for $1 \ll z \ll N^{\frac{4}{7-p}}$ and the string coupling will become constant for large

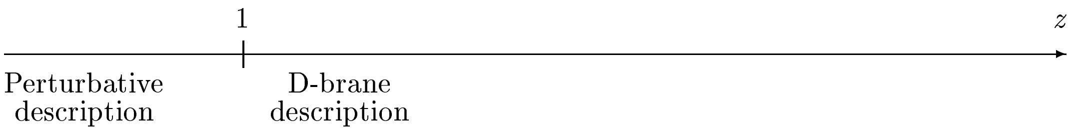

Figure 2: Phase diagram for the case where the D-brane description is valid for all $z \gg 1$.

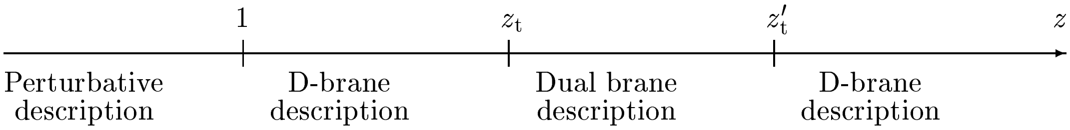

Figure 3: Phase diagram with two transition points z_t and z'_t where $g_s^{\mathrm{eff}} = 1$.

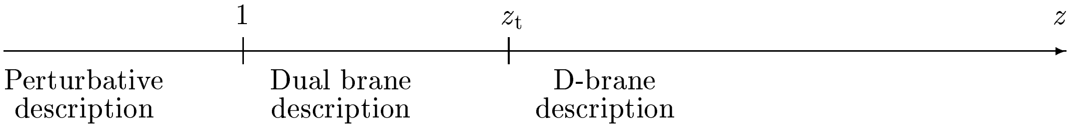

Figure 4: Phase diagram with one transition point z_t where $g_s^{\mathrm{eff}} = 1$ but with large g_s^{eff} for $z \ll z_\mathrm{t}$ and small g_s^{eff} for $z \gg z_\mathrm{t}$.

z. Except for the non-commutativity, the phase structure is the same as for $m = 0$, depicted in figure 1, with $z_\mathrm{t} \sim N^{\frac{4}{7-p}}$. We need $N \gg 1$.

Consider the case $z_\mathrm{nc} \ll N^{\frac{4}{7-p}}$. In this case we have that $g_s^{\mathrm{eff}} \ll 1$ for all $z \gg 1$. Thus, the supergravity Dp-brane description is valid for all $z \gg 1$. The phase diagram for this case in depicted in figure 2. If $z_\mathrm{nc} \ll 1$ it is possible to have N finite.

5. $\eta > \frac{7-p}{2m}$: here $z \gg z_\mathrm{nc}$ is the non-commutative sector, so for $z \ll 1$ we have a perturbative commutative SYM description (assuming that $z_\mathrm{nc} > 1$).

   Consider the case $N^{\frac{4}{7-p}} \ll z_\mathrm{nc}$. If $N \gg 1$, then it can be seen from (3.17) that the supergravity Dp-brane description is not only valid for $1 \ll z \ll N^{\frac{4}{7-p}}$ but also for $z \gg z'_\mathrm{t}$ where $z'_\mathrm{t} \gg z_\mathrm{nc}$. This interesting phase structure is depicted in the phase diagram in figure 3 with $z_\mathrm{t} \sim N^{\frac{4}{7-p}}$.

   In the case of finite N, the transition at $z = 1$ goes from a perturbative field theory description into a supergravity description with a brane configuration that is S-dual to the Dp-brane. At $z_\mathrm{t} \gg z_\mathrm{nc}$ we have a transition into a Dp-brane description. The corresponding phase diagram is depicted in figure 4.

   Consider the case $z_\mathrm{nc} \ll N^{\frac{4}{7-p}}$. In this case we have that $g_s^{\mathrm{eff}} \ll 1$ for all $z \gg 1$ so that the Dp-brane description is valid for all $z \gg 1$. The phase diagram

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corresponding to this case is depicted in figure 2. If $z_{\rm nc} \ll 1$ it is possible to have N finite.

From the above analysis we infer the following general conclusions:

(i) Comparing the analysis to the phase structure of the $m = 0$ case [5] a much richer structure is observed for $m > 0$: For $m = 0$ there was basically only one type of phase structure, namely the one depicted in figure 1. Instead, for $m > 0$ there are four types of phase diagrams, depicted in figures 1–4.

(ii) For each of the five cases, covering all values of η, it is possible to find a regime with finite N. In fact, this regime corresponds to having the non-commutativity as significant as possible, i.e. having the largest possible part of the $z = g_{\rm eff}^2$ phase space non-commutative. Thus, it is possible to have a dual supergravity description of NCSYM at strong coupling for finite N. It is interesting to note that the scaling factor N^2 in the extensive thermodynamic quantities (see section 4.1) persists for finite N. This is different from the strong coupling SYM at finite N, which has an $N^2 - 1$ factor instead, coming from the SU(N) group. In section 6 we comment on the connection between these two different factors.

(iii) The phase structure depicted in figure 2 is particularly interesting since only one phase is present for $z \gg 1$: the string coupling constant is small for all $z \gg 1$ so that the D-brane description is valid in this entire range. As one can extract from case 4 and 5 above, this phase structure occurs when $\eta \geq \frac{7-p}{2m}$ and $z_{\rm nc} \ll N^{\frac{4}{7-p}}$. If in addition $z_{\rm nc} \ll 1$ we can also have finite N.

We now consider some special choices of α, β, γ in (3.13), using eqs. (3.14) and (3.16). We are especially interested to find the cases that allow $\eta \geq \frac{7-p}{2m}$ since this gives rise to a very interesting phase structure.

a) $\alpha = \gamma = 0$: this case corresponds to the one described in ref. [10], where the energy r is varied while keeping all other parameters fixed. Clearly, we cannot have $p = 3$, and for the other branes we have $\beta = \frac{1}{p-3}$ and $\eta = \frac{7-p}{p-3}$. This means that $\eta \geq \frac{7-p}{2m}$ is equivalent to $p > 3$ and $p - 3 \leq 2m$. This is fulfilled for $p = 4, 5$ for $m \geq 1$ and $p = 6$ for $m \geq 2$, as was found in [10].

b) $\beta = \gamma = 0$: in this case only the YM coupling $g_{\rm YM}$ is varied. We have $\alpha = 1$ and $\eta = -1$, so that for all cases the phase structure is the one depicted in figure 1.

c) $\alpha = \beta = 0$: in this case we vary only the NC parameter b. We have $\gamma = \frac{1}{m}$ and $\eta = \frac{2-m}{m}$ so we have $\eta \geq \frac{7-p}{2m}$ only for $m = 1$ and $p = 5, 6$.

d) $\alpha + m\gamma = (5-p)\beta$: with this choice the quantity r^{5-p}/h^{7-p} is fixed which is necessary to keep the temperature T fixed, as we shall see in section 4.2. It follows that $\beta = 1/2$ and $\eta = 1 + 2\gamma$. Choosing $\gamma = 0$ (in order not to change the position commutators of the non-commutative field theory) we find that $\eta \geq \frac{7-p}{2m}$ for $p = 3, 4$ with $m = 2$ and $p = 5, 6$ with $m \geq 1$. The importance of this case will be clear in section 4.2 where it will be considered in relation with the thermodynamics.

We can also reverse the logic and ask whether it is possible for a given p and m to find α, β and γ such that a specific value of η is obtained. This is trivially seen to be true, in fact, since for a given η there are only two restrictions (3.14), (3.16) on the three scaling exponents α, β and γ, leaving a freedom of choice in these exponents. For example, with the additional constraint $\gamma = 0$, we find that $\alpha = 1 - (p-3)\frac{1+\eta}{4}$ and $\beta = \frac{1+\eta}{4}$. It is interesting to apply this to the D2-brane, since in [10] no deviation from the usual phase structure was found in this case. For $p = 2$ and $m = 1$ we have $\alpha = \frac{5+\eta}{4}$ and $\beta = \frac{1+\eta}{4}$ with the choice $\gamma = 0$. Thus, also for the D2-brane, all four types of phase structures are possible.

3.4 Non-zero angular momenta

In this section we extend the analysis of sections 3.2 and 3.3 to non-zero angular momenta l_i, $i = 1, \ldots, n$.

The isometry group $\mathrm{SO}(9-p)$ of the transverse sphere of a Dp-brane corresponds to the R-symmetry group of the dual field theory. From the point of view of the thermodynamics, the Cartan subgroup $\mathrm{SO}(2)^n$ of the $\mathrm{SO}(d)$ manifests itself as the thermodynamic quantities $\{\Omega_i\}$ corresponding to the angular velocities in supergravity and the R-voltages in the dual field theory, and $\{J_i\}$ which are the angular momenta in supergravity and the R-charges in the dual field theory (see e.g. [21, 22, 23, 24, 16]). The thermodynamics with non-zero l_i will be considered in section 4.

The effective field theory parameters g_{eff} and a_k^{eff}, $k = 1, \ldots, m$, are as before in (3.8) and (3.9) since r is still the Higgs expectation value of the brane probe. However, the effective dilaton is now(up to a constant)

$$g_s^{\mathrm{eff}} \sim \frac{g_{\mathrm{eff}}^{\frac{7-p}{4}} W_p^{\frac{p-3}{4}}}{N \prod_{k=1}^{m} (1 + W_p (a_k^{\mathrm{eff}})^{7-p})^{1/2}} , \tag{3.18}$$

which depends on both l_i, $i = 1, \ldots, n$, and the angles through the function W_p in (2.6).

If we consider the phase space parameterized by N, $g_{\rm YM}$, r, b and l_i, $i = 1, \dots, n$ we can investigate, as done in section 3.3 for zero l_i, the phase structure when varying $z = g_{\rm eff}^2$. From the metric (3.3a) and the effective dilaton (3.18) the deformations caused by the presence of non-zero angular momenta will depend on the ratios l_i/r, $i = 1, \dots, n$. If we parametrize, along with (3.13), the angular momenta as

$$l_i \propto z^\beta \,, \qquad i = 1, \dots, n \tag{3.19}$$

then the deformations arising from the angular momenta do not change with z.

The analysis of section 3.3 used two special calibration points, $z = 1$ where the curvature of the geometry is of order 1 and $z = z_{\rm t}$ where $g_s^{\rm eff} \sim 1$. It might seem that we cannot define these points anymore, since for a specific $g_{\rm YM}$, b and r we have a specific $z = g_{\rm eff}^2$, but both the curvature of the geometry and the effective dilaton field are clearly angular dependent. However, considering for definiteness the $z_{\rm t}$ point, we can define this instead by stating that for $z \ll z_{\rm t}$ we have $g_s^{\rm eff} \ll 1$ and $z \gg z_{\rm t}$ we have $g_s^{\rm eff} \gg 1$ (assuming that $g_s^{\rm eff}$ is increasing near $z_{\rm t}$). In other words, we only have to define points on the phase diagrams for z up to a certain order, defined by the large inequalities $\gg$ and $\ll$. Therefore, for a given set of l_i/r we can choose the scale of the variables N, $g_{\rm YM}$, b, r and l_i, so that the phase transition points in the phase diagram are defined with an accuracy good enough to have regions with distinct phases. In this sense, the ratios l_i/r, $i = 1, \dots, n$ can be of any order, so long as we choose the phase transition points to be of high enough order for them to be well defined. Hence, non-zero angular momenta do not induce any modifications to the gauge coupling phase structure found in section 3.3.

4. Thermodynamics of NCSYM from supergravity

4.1 The thermodynamic quantities

In section 2 it was shown that the thermodynamics of a spinning D-brane bound state is essentially the same as that of the spinning Dp-brane, with p being the spatial dimension of the D-brane bound state. The only change is the appearance of extra charges and chemical potentials, but the temperature T, entropy S, angular velocity Ω_i and angular momentum J_i were unchanged. We now consider the thermodynamics of the near-horizon solution given in section 3.1, which can be obtained from (2.11) using the rescalings (3.1) and (3.2). We also note that the energy is computed from the energy above extremality $E = M - (\sum Q^2)^{1/2}$ in the near-horizon limit, where the sum is over all charges in the bound state. Here, it is used that at extremality the bound state is a 1/2-BPS state, and we recall that the expressions of the charges and chemical potentials are given in appendix B.

One then finds that the charges as well as the chemical potentials are constant in the near-horizon limit, and hence do not appear in the thermodynamics. For the remaining thermodynamic quantities we find in the near-horizon limit

$$T = \frac{w_p}{4\pi}\lambda^{-1/2}(7-p-2\kappa)\frac{r_0^{\frac{7-p}{2}}}{r_H}\,, \qquad S = 4\pi\hat{w}_p V_p N^2 \lambda^{-3/2} r_0^{\frac{7-p}{2}} r_H \tag{4.1a}$$

$$\Omega_i = w_p \lambda^{-1/2} r_0^{\frac{7-p}{2}} \frac{l_i}{l_i^2 + r_H^2}\,, \qquad J_i = 2\hat{w}_p V_p N^2 \lambda^{-3/2} r_0^{\frac{7-p}{2}} l_i \tag{4.1b}$$

$$E = \frac{9-p}{2}\tilde{w}_p V_p N^2 \lambda^{-2} r_0^{7-p}\,, \tag{4.1c}$$

where we have defined the NC 't Hooft coupling,

$$\lambda = g_{\rm YM}^2 N \prod_{k=1}^{m} b_k \tag{4.2}$$

and where

$$w_p = \sqrt{\frac{(7-p)V(S^{8-p})}{(2\pi)^{9-2p}}}\,, \qquad \hat{w}_p = (2\pi)^{2p-11}\sqrt{\frac{(2\pi)^{9-2p}V(S^{8-p})}{7-p}}$$
$$\tilde{w}_p = w_p\hat{w}_p = (2\pi)^{2p-11}V(S^{8-p})\,. \tag{4.3}$$

It is not difficult to verify that the energy satisfies the first law of thermodynamics

$$dE = TdS + \sum_{i=1}^{n}\Omega_i dJ_i\,, \qquad E = E(S,\{J_i\}) \tag{4.4}$$

and the integrated Smarr formula [16]

$$(7-p)E = \frac{9-p}{2}TS + \frac{9-p}{2}\sum_{i=1}^{n}\Omega_i J_i\,. \tag{4.5}$$

Comparing the thermodynamics (4.1) with the one obtained for the near-horizon spinning Dp-brane in ref. [16] we observe that they are identical up to the replacement

$$g_{\rm YM}^2 \to g_{\rm YM}^2 \prod_{k=1}^{m} b_k\,, \tag{4.6}$$

where $g_{\rm YM}$ on the left-hand side is the YM coupling constant of the commutative theory and $g_{\rm YM}$ on the right-hand side of the non-commutative theory. For the non-rotating case this was found in [7, 10, 12, 13] and argued at weak coupling from the field theory point of view in [11].

For various applications it is also useful to compute the Gibbs free energy F from eqs. (4.1), with the result

$$F = E - TS - \sum_{i=1}^{n}\Omega_i J_i = -\frac{5-p}{9-p}E = -\frac{5-p}{2}\tilde{w}_p V_p N^2 \lambda^{-2} r_0^{7-p} \tag{4.7}$$

satisfying

$$dF = -SdT - \sum_{i=1}^{n} J_i d\Omega_i \,, \qquad F = F(T, \{\Omega_i\}) \tag{4.8}$$

For $p = 2, 3, 4$ and one non-zero angular momentum the exact form of the free energy in terms of the intensive thermodynamic quantities is [16]

$$F(T,\Omega) = -c_p V_p N^2 \lambda^{-\frac{p-3}{p-5}} T^{\frac{2(7-p)}{5-p}} (1+x)^{2(6-p)/(5-p)} \left(1 + \frac{x}{x_c}\right)^{-2\frac{7-p}{5-p}} \tag{4.9a}$$

$$x \equiv \frac{l^2}{r_H^2} = 8 \left(\frac{7-p}{4\pi}\right)^2 \frac{\omega_c^4}{\omega^2} \left(1 - \frac{1}{2}\left(\frac{\omega}{\omega_c}\right)^2 - \sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}\right), \quad \omega \equiv \frac{\Omega}{T}\,, \tag{4.9b}$$

where the constants c_p can be found in ref. [16] and

$$x_c = \frac{7-p}{5-p}\,, \qquad \omega_c = \frac{2\pi}{\sqrt{(7-p)(5-p)}}\,. \tag{4.10}$$

The boundaries of stability and critical exponents for these cases can also be found in ref. [16], and we note in particular the stability bound $2\pi J/S \leq \sqrt{x_c}$, with the boundary of stability at $\omega = \omega_c$. For more than one angular momentum the free energy has the same prefactor as in (4.9) and the expansion in powers of ω including the first two subleading terms can be found in [16]. For the case $p = 5$ the free energy is zero since T and Ω_i are not independent variables anymore. For the corresponding results in the D6-brane case we refer to section 4.3, which includes a more detailed account of the thermodynamics as it shows some exotic behavior and will be relevant for the 7-dimensional non-commutative field theory that is dual to this supergravity solution.

4.2 Ranges of validity

In this section we discuss the range of validity in which the thermodynamics of the near-horizon limit (see section 4.1) describes the thermodynamics of the NCSYM on the world-volume of the D-brane. In particular, we supplement the analysis of the gauge coupling phase structure of section 3.3 with further details on the gauge coupling phase space relevant for the special case in which the effective gauge coupling can vary independently of the near-horizon intensive thermodynamic quantities (case d of section 3.3). Specifically, we explain the cases for which i) the coupling can go all the way to infinity, and ii) finite N is allowed, and still have a valid thermodynamic description.

The criterion that we impose for invariant thermodynamics is that intensive, scale-independent quantities like the temperature T and the R-voltages Ω_i must remain fixed, while the extensive quantities like the entropy S and the R-charges J_i are allowed to change when all the extensive quantities change uniformly. As discussed in sections 3.3 and 3.4, for fixed N the gauge theory phase space is characterized by variables $g_{\rm YM}$, r_0, l_i, b, and the path in the gauge theory phase space is parameterized by

$$g_{\rm YM}^2 \propto z^{\alpha}\,, \qquad r_0 \propto z^{\beta}\,, \qquad l_i \propto z^{\beta}\,, \qquad b \propto z^{\gamma}\,, \tag{4.11}$$

where $z = g_{\rm eff}^2 = g_{\rm YM}^2 N b^m r_H^{p-3}$. It is trivial to see that $r_H \propto z^{\beta}$. Note that we use r_H here rather than r in $g_{\rm eff}$ since the validity of the D-brane thermodynamics is considered. From the intensive quantities in (4.1) it then follows that we need to keep $r_0^{5-p}\lambda^{-1}$ fixed, yielding the restriction

$$\alpha + m\gamma = (5-p)\beta\,. \tag{4.12}$$

As shown in section 3.3, it follows from (3.14) and (4.12) that $\beta = 1/2$ and $\eta = 1+2\gamma$. The scaling of the extensive quantities is then proportional to $z^{\frac{p-3}{2}}$ under the variation of z.

We choose $\gamma = 0$ in the following, which seems most natural considering that the position commutators for the non-commutative coordinates are proportional to b [7, 10]. With this choice, $\eta = 1$, so that the non-commutative sector is $z \gg z_{\rm nc}$ while the commutative sector is $z \ll z_{\rm nc}$. In the regime $z_{\rm nc} \ll 1$, N can be finite. As noted in section 3.3, the condition $\eta \geq \frac{7-p}{2m}$ is fulfilled for $p = 4$ with $m = 2$ and $p = 5, 6$ with $m \geq 1$. In these cases, if also $z_{\rm nc} \ll N^{\frac{4}{7-p}}$ then the effective coupling $z = g_{\rm eff}^2$ can go all the way to infinity with the D-brane thermodynamics being valid[6] for the dual field theory, as depicted in the phase diagram of figure 2.

From the above considerations, we see that the most interesting part of parameter space is the region where $z_{\rm nc} \ll 1$, since here both N can be finite, and in the indicated cases we also have that the D-brane thermodynamics is valid for the dual field theory for as large coupling as desired. It is interesting to note that the requirement $z_{\rm nc} \ll 1$ can be seen as demanding the non-commutativity effects to be as significant as possible. We therefore find this region expressed in terms of the intensive thermodynamic parameters T and Ω_i. Since $\beta = 1/2$ we have from (4.11),

$$r_0 = \hat{r}_0\sqrt{z}\,, \qquad r_H = \hat{r}_H\sqrt{z}\,, \qquad l_i = \hat{l}_i\sqrt{z}\,, \tag{4.13}$$

with the parameters $\hat{r}_0$, $\hat{r}_H$ and $\hat{l}_i$ invariant under variation of z. With this, the invariant part of λ is $\hat{\lambda} = \hat{r}_H^{3-p}$. Using the definition (3.15) of $z_{\rm nc}$ it follows after some substitutions that,

$$z_{\rm nc} \sim b^{-2}\hat{r}_H^{-4} \tag{4.14}$$

[6]For the D3 and D5-brane this holds for all m, as explained in section 3.2.

so that

$$z_{\rm nc} \ll 1 \;\Leftrightarrow\; \hat{r}_H \gg b^{-1/2} \,. \tag{4.15}$$

This corresponds to the UV region, as can be seen by writing $E = \hat{E} z^{\frac{p-3}{2}}$, where the z-independent part of the energy $\hat{E}$ is of the same order as $\hat{r}_0^{7-p} \hat{r}_H^{2p-6}$ so that $\hat{E} \gg b^{-\frac{p+1}{2}}$ since $\hat{r}_0 \geq \hat{r}_H$. This is in agreement with the fact that non-commutative effects are expected to be of significance in the UV-region.

Substituting (4.13) in (4.1) we obtain for the intensive thermodynamic quantities

$$T \sim \hat{r}_H \left(\frac{\hat{r}_0}{\hat{r}_H} \right)^{\frac{7-p}{2}} (7 - p - 2\kappa) \,, \qquad \Omega_i \sim \hat{r}_H^2 \left(\frac{\hat{r}_0}{\hat{r}_H} \right)^{\frac{7-p}{2}} \frac{\hat{l}_i}{\hat{l}_i^2 + \hat{r}_H^2} \,. \tag{4.16}$$

Provided we are away from the region in which κ is near $\frac{7-p}{2}$, we then have

$$z_{\rm nc} \ll 1 \;\Leftrightarrow\; T \gg b^{-1/2} \left(\frac{\hat{r}_0}{\hat{r}_H} \right)^{\frac{7-p}{2}} \tag{4.17}$$

showing that we are in the high-temperature region of the $(T, \{\Omega_i\})$ phase space. If we have that $\hat{l}_i/\hat{r}_0$ is of order one or less, then $\hat{r}_H \sim \hat{r}_0$ and (4.17) reduces to

$$z_{\rm nc} \ll 1 \;\Leftrightarrow\; T \gg b^{-1/2} \,. \tag{4.18}$$

For large angular momenta we have that $\hat{r}_H \ll \hat{r}_0$, in which case even larger temperatures are necessary.

If κ (see eq. (2.13)) is very near $\frac{7-p}{2}$ it follows from (4.16) that low temperatures are allowed as well. For $\kappa = \frac{7-p}{2}$ we have $T = 0$, but we can clearly still have $\hat{r}_H \gg b^{-1/2}$. It is not clear what the significance of this special region of the $(T, \{\Omega_i\})$ phase space is, but we note that this region is also of interest because the temperature can be zero with the other thermodynamic quantities being non-zero.

4.3 Thermodynamics of the D6-brane theory with B-field

In this section we analyze the thermodynamics of the near-horizon spinning D6-brane with a B-field, reviewing and adding to the analysis of ref. [16]. This theory is of interest since it has been shown [7, 10] that the D6-brane with non-zero B-field decouples from gravity in the near-horizon limit. This suggests the existence of a consistent 7-dimensional NCSYM, dual to the near-horizon D6-brane solution. We can thus obtain information about the thermodynamics of this 7-dimensional NCSYM with R-voltage and R-charge by considering the spinning near-horizon D6-brane. As we shall see, the D6-brane theory has various exotic features distinguishing it from the other Dp-brane theories.

Substituting $p = 6$ in the general formulae (4.1) and (4.7) one obtains the thermodynamic quantities

$$T = \sqrt{2}\pi\lambda^{-1/2}\frac{1 - l^2/r_H^2}{\sqrt{r_0}}\,, \qquad S = 4\sqrt{2}\pi V_6 N^2 \lambda^{-3/2}\sqrt{r_0}r_H \tag{4.19a}$$

$$\Omega = 4\sqrt{2}\pi^2\lambda^{-1/2}\frac{l}{r_H}\frac{1}{\sqrt{r_0}}\,, \qquad J = 2\sqrt{2}V_6 N^2\lambda^{-3/2}\sqrt{r_0}l \tag{4.19b}$$

$$E = 12\pi^2 V_6 N^2\lambda^{-2}r_0\,, \qquad F = 4\pi^2 V_6 N^2 \lambda^{-2} r_0\,, \tag{4.19c}$$

where the horizon radius is determined by (2.13) as

$$r_H = \frac{r_0}{2}\left(1 + \sqrt{1 - \frac{4l^2}{r_0^2}}\right). \tag{4.20}$$

This means that the angular momentum and horizon radius are restricted to the ranges $0 \leq l \leq r_0/2$ and $r_0/2 \leq r_H \leq r_0$ respectively. From this we see that $l \leq r_H$ so that it follows from (4.19a) that $T \geq 0$ and that $T = 0$ for $l = r_0/2 = r_H$.

We now address the question of thermodynamic stability of the D6-brane theory. We begin by considering the grand canonical ensemble, in which the system is in contact with a heat reservoir of temperature T and R-voltage Ω. Defining $\tilde{S} = \sqrt{r_0}r_H$, $\tilde{J} = \sqrt{r_0}l$ and $\tilde{E} = r_0$, which are rescaled versions of S, J and E in (4.19), we find the relation

$$\tilde{S} = \frac{1}{2}\left(\tilde{E}^{3/2} + \sqrt{\tilde{E}^3 - 4\tilde{J}^2}\right). \tag{4.21}$$

From (4.21) one can check that the Hessian of the entropy always has one negative and one positive eigenvalue, and the theory is therefore thermodynamically unstable in the entire (E, J) phase space.

Another way to arrive at this result is by considering the Gibbs free energy

$$F(T, \Omega) = 8\pi^4 V_6 N^2 \lambda^{-3} T^{-2} \xi\left((2\pi)^{-2}\omega^2\right), \tag{4.22}$$

where $\omega = \Omega/T$ and ξ is the function defined by

$$\xi(x) = -\frac{8}{x^2}\left(\sqrt{1+x} - 1 - \frac{1}{2}x\right) = 1 - \frac{1}{2}x + \frac{5}{16}x^2 - \frac{7}{32}x^3 + \mathcal{O}(x^4) \tag{4.23}$$

for $x \geq -1$. Note that the Gibbs free energy is positive, contrary to the Gibbs free energies in (4.9) for the other near-horizon Dp-brane theories. The Gibbs free energy is a function on the (T, Ω) phase space, the properties of which depend on the map between the supergravity variables (r_H, l) and (T, Ω). For the D6-brane this map is one-to-one, which is a consequence of the fact that the determinant of the Hessian of the Gibbs free energy [16]

$$\det \mathrm{Hes}(F) = -6\pi^{-2} V_6^2 N^4 \lambda^{-2} r_0^4 \left(1 + \frac{l^2}{r_H^2}\right)^{-3} \tag{4.24}$$

is neither zero nor singular. Since this determinant is always negative, the theory is clearly unstable for all points in the (T, Ω) phase space. We remark that because the map between (r_H, l) (or equivalently (S, J)) and (T, Ω) is one-to-one, the phase mixing mechanism argued by Cvetic and Gubser in [24] for the D3-brane, is impossible for the D6-brane theory.

Turning instead to the canonical ensemble, in which case the system is in contact with a heat bath of temperature T at a fixed R-charge J, the heat capacity takes the form [16]

$$C_J = 12\sqrt{2}\pi V_6 N^2 \lambda^{-3/2} \sqrt{r_0} r_H \frac{(r_H^2 - l^2)(r_H^2 + l^2)}{5l^4 + 8l^2 r_H^2 - r_H^4} \,. \tag{4.25}$$

From this expression it follows that C_J is negative for $0 \leq l^2/r_H^2 < \frac{\sqrt{21}-4}{5}$, zero at $l^2/r_H^2 = \frac{\sqrt{21}-4}{5}$, positive for $\frac{\sqrt{21}-4}{5} < l^2/r_H^2 < 1$ and zero again at $l^2/r_H^2 = 1$. The D6-brane theory thus has the remarkable property that it is thermodynamically stable in the canonical ensemble, in the range $\frac{\sqrt{21}-4}{5} \leq l^2/r_H^2 \leq 1$ [16]. This can also be written as $\omega = \Omega/T \geq \omega_c$ with $\omega_c \simeq 4.8551$. In [16] the weakly-coupled D6-brane theory was studied in the ideal gas approximation where it was found to exhibit a qualitatively different behaviour, with the brane being stable when $\omega \leq \omega_c' \simeq 4.9948$. The weak and strong coupling limits of this theory should therefore be connected via a rather non-trivial phase transition.

If we consider the euclidean version of the D6-brane theory, we have to perform the Wick rotation $\tau = it$ and $\tilde{l} = -il$. This yields the restriction $0 \leq \tilde{l}^2/r_H^2 < 1$ for the euclidean theory, since $r_0 = (1 - \tilde{l}^2/r_H^2) r_H$. When $\tilde{l}^2/r_H^2 \to 1$ the supersymmetry of the theory is restored, since the supersymmetric boundary conditions are recovered from the R-symmetry group element in the partition function [24].

The corresponding expression for the entropy is obtained with the substitution $\tilde{J}^2 \to -\tilde{J}^2$ in (4.21), and it is easy to check that the Hessian of the entropy always has two negative eigenvalues. Thus, the euclidean D6-brane theory is also thermodynamically unstable in the (E, J) phase space, so there are no stable regions in the grand canonical ensemble. If we consider the canonical ensemble, it is easily seen that the heat capacity (4.25) is never positive, and hence there are no stable regions in this ensemble either. The limit $\tilde{l}^2/r_H^2 \to 1$ appears to be singular as, for example, the temperature diverges in this limit. The issue of stability is not clear, since the second derivatives of the entropy or the free energies go to zero or infinity. It is thus unclear, and highly questionable, whether a sensible theory can be recovered in this limit.

5. Spinning M-brane bound states and non-commutative $(2,0)$ theory

5.1 Spinning M5-M2 brane bound state

The spinning M5-M2 brane bound state can be obtained from the spinning D4-D2 brane bound state ($p = 4$, $m = 1$) of section 2 by lifting to M-theory, which gives the metric

$$ds^2 = (HD)^{-1/3} \Bigg[-f_5 dt^2 + (dy^1)^2 + (dy^2)^2 + D\Big((dy^3)^2 + (dy^4)^2 + (dy^5)^2\Big) + \\ + H\Big(\bar{f}_5^{-1} K_5 dr^2 + \Lambda_{\alpha\beta} d\eta^\alpha d\eta^\beta\Big) + \\ + \frac{1}{K_5 L_5} \frac{r_0^3}{r^3} \left(\sum_{i,j=1}^{2} l_i l_j \mu_i^2 \mu_j^2 d\phi_i d\phi_j - 2\cosh\alpha \sum_{i=1}^{2} l_i \mu_i^2 dt d\phi_i \right) \Bigg] \quad (5.1)$$

and gauge potentials

$$\begin{aligned} \mathcal{C}_3 &= -\sin\theta (H^{-1} - 1) I \wedge dy^1 \wedge dy^2 + \tan\theta D H^{-1} dy^3 \wedge dy^4 \wedge dy^5 \\ \mathcal{E}_6 &= \cos\theta D (H^{-1} - 1) I \wedge dy^1 \wedge \cdots \wedge dy^5 \,. \end{aligned} \quad (5.2)$$

Here, the functions f_5, $\bar{f}_5$, K_5 and L_5 are as in (2.5) with $p = 4$, D is defined in (2.8) and the one-form I is defined in (B.1). We also need the relations

$$16\pi G = (2\pi)^8 l_p^9 \,, \qquad h^3 = \frac{\hat{h}^3}{\cos\theta} \,, \qquad \hat{h}^3 = \pi N l_p^3 \,. \quad (5.3)$$

where l_p is the 11-dimensional Planck length, h is defined in (2.9) (with $p = 4$) and the second relation follows from charge quantization of the M5-brane. For zero angular momentum the solution reduces to the one given in ref. [18].

Aside from the presence of charges and chemical potentials for both the M5 and M2-brane, the thermodynamic quantities of this background are not altered due to the presence of D, and coincide with those of the spinning M5-brane. For example, it is not difficult to see that the temperature is not modified by the presence of D and also cancels out in the horizon area (and hence the entropy); the same conclusion holds for the mass, angular momentum and velocity. The complete set of thermodynamic quantities is then given by

$$M = \frac{V_5 V(S^4)}{16\pi G} r_0^3 \Big[4 + 3\sinh^2\alpha\Big] \quad (5.4a)$$

$$T = \frac{3 - 2\kappa}{4\pi r_H \cosh\alpha} \,, \qquad S = \frac{V_5 V(S^4)}{4G} r_0^3 r_H \cosh\alpha \quad (5.4b)$$

$$\Omega_i = \frac{l_i}{(l_i^2 + r_H^2)\cosh\alpha} \,, \qquad J_i = \frac{V_5 V(S^4)}{8\pi G} r_0^3 l_i \cosh\alpha \quad (5.4c)$$

$$\mu_5 = \cos\theta\mu \,, \qquad \mu_2 = -\sin\theta\mu \,, \qquad Q_5 = \cos\theta Q \,, \qquad Q_2 = -\sin\theta Q \quad (5.4d)$$

$$\mu = \tanh\alpha \,, \qquad Q = \frac{V_5 V(S^4)}{16\pi G} 3 r_0^3 \sinh\alpha\cosh\alpha \quad (5.4e)$$

satisfying the integrated Smarr formula

$$3M = 4TS + 3(\mu_5 Q_5 + \mu_2 Q_2) + 4\sum_{i=1}^{2} \Omega_i J_i \tag{5.5}$$

which is a consequence of the first law of thermodynamics (2.14).

5.2 The near-horizon limit

Next we turn to the near-horizon limit of the spinning M5-M2 brane bound state, which corresponds [7, 2, 10] to the six-dimensional non-commutative (2,0) theory [25]. The near-horizon limit is defined by letting the Planck length $l_p \to 0$ accompanied by the rescalings[7]

$$r = \frac{r_{\rm old}}{l_p^3}\,, \qquad r_0 = \frac{(r_0)_{\rm old}}{l_p^3}\,, \qquad l_i = \frac{(l_i)_{\rm old}}{l_p^3} \tag{5.6a}$$

$$\hat{h}^3 = \frac{\hat{h}^3_{\rm old}}{l_p^3}\,, \qquad c = l_p^6 \tan\theta \tag{5.6b}$$

$$y_i = \frac{l_p^3}{c^{1/2}}(y_i)_{\rm old}\,, \quad i = 0,1,2\,, \qquad y_i = \frac{c^{1/2}}{l_p^3}(y_i)_{\rm old}\,, \quad i = 3,4,5 \tag{5.6c}$$

$$ds^2 = \frac{ds^2_{\rm old}}{l_p^2}\,, \qquad \mathcal{C}_3 = \frac{(\mathcal{C}_3)_{\rm old}}{l_p^3}\,, \qquad \mathcal{E}_6 = \frac{(\mathcal{E}_6)_{\rm old}}{l_p^6}\,, \qquad G = \frac{G_{\rm old}}{l_p^9} \tag{5.6d}$$

where the quantities on the left hand side are kept fixed and the quantities on the right-hand side (labelled with subscript "old" except for θ) are those that enter the asymptotically-flat solution (5.1), (5.2). Note that the rescaling in (5.6d) leaves the 11-dimensional low-energy supergravity action invariant. It also follows that $e^{2\alpha} \to 4c(\hat{h}/r_0)^3 l_p^{-12}$, which is used below.

The metric of the resulting near-horizon background of the spinning M5-brane with non-zero $\mathcal{C}_3$-field is then given by

$$\begin{aligned} ds^2 = (HD)^{-1/3}\Big[& -f_5 dt^2 + (dy^1)^2 + (dy^2)^2 + D[(dy^3)^2 + (dy^4)^2 + (dy^5)^2] + \\ & + H\Big(\bar{f}_5^{-1} K_5 dr^2 + \Lambda_{\alpha\beta} d\eta^\alpha d\eta^\beta\Big) - 2\frac{1}{K_5 L_5}\frac{(r_0\hat{h})^{3/2}}{r^3}\sum_{i=1}^{2} l_i \mu_i^2 dt d\phi_i \Big]\,, \end{aligned} \tag{5.7}$$

where

$$H = \frac{\hat{h}^3}{K_5 L_5 r^3}\,, \qquad D = [1 + a^3 K_5 L_5 r^3]^{-1}\,, \qquad a^3 = \frac{c}{\hat{h}^3} \tag{5.8}$$

[7] Note that r corresponds to energy squared in our conventions.

and $\hat{h}^3 = \pi N$ after the rescaling. The expressions for the rescaled 3-form potential is likewise given by

$$\mathcal{C}_3 = -c\left(H^{-1}dt + \left(\frac{r_0}{\hat{h}}\right)^{3/2}\sum_{i=1}^{2} l_i \mu_i^2 d\phi_i\right) \wedge dy^1 \wedge dy^2 + \frac{1}{c^2}H^{-1}Ddy^3 \wedge dy^4 \wedge dy^5\,, \tag{5.9}$$

where we have ignored a constant term that does not contribute to the field strength. The expression for the potential $\mathcal{E}_6^{\rm tot} = \mathcal{E}_6 - \frac{1}{2}\mathcal{C}_3^2$ entering the 7-form field strength can be obtained in the same way, but is a bit more involved.

To compute the thermodynamic quantities in the near-horizon limit, one may use the quantities (5.4) of the asymptotically-flat solution along with the rescaling (5.6), and employ the formula $E = M - (\sum Q^2)^{1/2}$ for the internal energy. However, since the time coordinate is also rescaled, we also have that

$$E = E_{\rm old}\frac{c^{1/2}}{l_p^3}\,, \qquad T = T_{\rm old}\frac{c^{1/2}}{l_p^3}\,, \qquad \Omega_i = (\Omega_i)_{\rm old}\frac{c^{1/2}}{l_p^3} \tag{5.10}$$

so that the following thermodynamics is obtained in the near-horizon limit

$$E = \frac{5}{3}\frac{V_5}{(2\pi)^6}r_0^3 \tag{5.11a}$$

$$T = \frac{3-2\kappa}{4\pi r_H}\frac{r_0^{3/2}}{(\pi N)^{1/2}}\,, \qquad S = \frac{4}{3}\frac{V_5(\pi N)^{1/2}}{(2\pi)^5}r_0^{3/2}r_H \tag{5.11b}$$

$$\Omega_i = \frac{l_i}{(l_i^2 + r_H^2)}\frac{r_0^{3/2}}{(\pi N)^{1/2}}\,, \qquad J_i = \frac{4}{3}\frac{V_5(\pi N)^{1/2}}{(2\pi)^6}r_0^{3/2}l_i \tag{5.11c}$$

satisfying the near-horizon Smarr formula [16]

$$3E = \frac{5}{2}TS + \frac{5}{2}\sum_{i=1}^{2}\Omega_i J_i\,, \tag{5.12}$$

which is a consequence of the first law of thermodynamics (4.4).

Eq. (5.11) describes the thermodynamics of the non-commutative (2,0) theory, and is manifestly independent of c, i.e. coincides with that of commutative (2,0) theory. Just as for the Dp-brane case, the only difference of the presence of the non-zero $\mathcal{C}_3$ potential is that the range of validity can be different now. In fact, focusing on zero angular momentum first, the curvature is [10]

$$R \sim \left(N^{2/3}(1 + a^3 r^3)\right)^{-1}\,, \tag{5.13}$$

which needs to be small in order to trust the supergravity description. This can be achieved either in the limit $N \gg 1$ as in the commutative case, but now there is the additional possibility of keeping N finite and requiring $r \gg 1/a$. This shows that

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the larger the non-commutative parameter a, the larger the energy range in which the near-horizon limit is a valid description of the non-commutative $(2,0)$ theory. Following a similar reasoning as in section 3.4 this conclusion holds also when the angular momenta are non-zero.

Finally, the general formulae of ref. [16] may be used to compute e.g. the internal energy and Gibbs free energy in terms of the extensive or intensive thermodynamic quantities respectively. For example, for one non-zero angular momentum the free energy takes the form

$$F(T,\Omega) = -\frac{2^6\pi^3}{3^7} N^3 V_5 T^6 (1+x)^4 \left(1+\frac{x}{x_c}\right)^{-6} , \tag{5.14}$$

where x, x_c are given by the $p=4$ expressions in (4.9b), (4.10). For the resulting boundaries of stability and critical exponents of the spinning M5-brane see refs. [23, 24, 16].

6. Conclusions and discussion

In this paper we have constructed general spinning brane bound state solutions of string and M-theory, and discussed their thermodynamics, extending our work [16]. Except for additional charges and chemical potentials of the lower branes in the bound state, the thermodynamics is equivalent to that of the highest brane in the bound state. We have computed the near-horizon limits of these supergravity solutions, which are dual to NCSYM theories for string theory and the non-commutative (2,0) theory for M-theory. Using this correspondence, we have presented a general analysis of the gauge coupling phase structure of these NCSYM theories by considering a general path in their phase space, both for zero as well as non-zero angular momenta. This analysis, which includes the one in ref. [10] as a special case, exhibits various interesting features, including regions in which the supergravity description is valid for finite N and/or the effective coupling ranging from the transition point $g_{\rm eff} \sim 1$ all the way to infinity. More generally, four types of phase diagrams are found, and we have established that for each spatial worldvolume dimension of the brane and each non-zero rank of B-field, a path and region of phase space can be chosen such that the phase structure of any of the four phase diagrams can be realized.

The thermodynamics of the near-horizon solutions is not altered by the presence of the B-field, in parallel with results for the non-rotating case [7, 10, 12, 13] showing that, to leading order, the thermodynamics of SYM is equivalent to that of NCSYM. This was argued at weak coupling from the field theory point of view [11] by showing that the planar limit of SYM and NCSYM coincide. As an application of the general phase structure analysis, the validity of the thermodynamics for the NCSYM has been examined by requiring that the intensive thermodynamic parameters are invariant for the path in phase space. We have determined the region of phase space in which

N can be finite and at the same time the coupling can be taken all the way to infinity. The resulting condition is that

$$T \gg b^{-1/2} \left(\frac{\hat{r}_0}{\hat{r}_H} \right)^{\frac{7-p}{2}} \tag{6.1}$$

showing that that at fixed non-extremality parameter $\hat{r}_0$ and horizon radius $\hat{r}_H$, the larger the non-commutativity parameter b, the larger the temperature region in which these properties are satisfied. Interestingly, for κ near $\frac{7-p}{2}$ low temperatures are allowed as well.

Having non-zero angular momenta does not qualitatively change the gauge coupling phase structure, but may well provide further insights into NCSYM in the presence of voltages for the R-charges. Moreover, as another application we have discussed the D6-brane theory in further detail, adding to the results of [16]. This theory is of interest in view of the recent discovery that the D6-brane theory with B-field decouples from gravity [7, 10]. Moreover, while the non-rotating case is thermodynamically unstable, for the spinning D6-brane stability is found in the canonical ensemble for sufficiently high angular momentum density.[8]

Unlike in the commutative case, the non-commutative setup allows for situations in which the supergravity description is valid for finite N. While in the usual large-N limit the $1/N$ corrections[9] around the planar limit are generated by the string loop expansion, for finite N this cannot be true anymore. Indeed, it is not difficult to see that in this case the effective string coupling (3.10) is small not because N is large, but rather since the effective NC parameter a^{eff} is large, so that the string loop expansion becomes an expansion in $1/a^{\text{eff}}$. This indicates that, in some sense, the framework enables one to interchange the large-N expansion with a large a^{eff} expansion. We also note that the α' expansion of higher derivative terms in the effective action, will generate $1/\lambda$ corrections,[10] with λ being the NC 't Hooft coupling (4.2).

The argument above provides insight into the issue raised in section 3.3: while the supergravity description for SYM in the strong coupling and large-N limit gives extensive thermodynamic quantities proportional to N^2, at weak coupling and finite N these scale with N^2–1, coming from the SU(N) group. In the commutative AdS/CFT correspondence for N D3-branes it is believed [27] that string loop corrections scaling as $1/N^2$, will generate the correct N^2-1 factor, in accord with strong/weak coupling duality of $N = 4$ SYM. On the other hand, for strongly coupled NCSYM at finite N (and thus large a^{eff}) the extensive thermodynamic quantities are proportional to N^2. However, as argued above, in this case the $1/a^{\text{eff}}$ corrections should generate the desired correction to N^2 to obtain N^2-1, thereby connecting NCSYM to SYM. We finally note that the N^2 dependence is in agreement with the observation that the non-commutative field theory at weak coupling has gauge group U(N) [2].

[8]We note that the D6-brane theory is also related to M(atrix) theory on T^6 [26].

[9]See ref. [12] for a discussion of $1/N$ corrections to the entropy of NCSYM.

[10]For the D3-brane case, tree-level R^4 corrections have been recently addressed in [13].

Note added After completion of this paper, the interchange of the large-N with large $a^{\rm eff}$ expansion was also found from a field theoretic analysis at weak coupling by demonstrating that only planar diagrams contribute in the maximally noncommutative limit $a^{\rm eff} \to \infty$ [28].

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A. T-duality as a solution generator

The spinning bound state solution (2.2)–(2.4) (and RR potentials given in appendix B) can be obtained from the general spinning Dp-brane solution of [16] by repeated use of the following sequence of T-dualities and coordinate transformations [18, 19]. For each disjoint pair of spatial dimensions in the world-volume of the Dp-brane the relevant part of the background is

$$\begin{aligned} & ds^2 = \sum_{i,j=1}^{2} g_{ij} dy^i dy^j\,, \qquad g_{11} = g_{22} = H^{-1/2}\,, \qquad g_{12} = 0 \\ & B_{12} = 0\,, \qquad e^{2\phi} = H^{(3-p)/2} \end{aligned} \tag{A.1}$$

Performing a T-duality transformation in the y^2-direction we obtain

$$\begin{aligned} & ds^2 = \sum_{i,j=1}^{2} {g'}_{ij} dy^i dy^j\,, \qquad {g'}_{11} = H^{-1/2}\,, \qquad {g'}_{22} = H^{1/2} \\ & {B'}_{12} = 0\,, \qquad e^{2\phi'} = e^{2\phi} H^{1/2} \end{aligned} \tag{A.2}$$

Next, one rotates the coordinates

$$\begin{pmatrix} \tilde{y}^1 \\ \tilde{y}^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \tag{A.3}$$

yielding the background

$$ds^2 = \sum_{i,j=1}^{2} \tilde{g}_{ij} d\tilde{y}^i d\tilde{y}^j \tag{A.4a}$$

$$\tilde{g}_{11} = H^{-1/2}\cos^2\theta + H^{1/2}\sin^2\theta \tag{A.4b}$$

$$\tilde{g}_{22} = H^{-1/2}\sin^2\theta + H^{1/2}\cos^2\theta = H^{1/2} D^{-1} \tag{A.4c}$$

$$\tilde{g}_{12} = \sin\theta\cos\theta \Big(H^{1/2} - H^{-1/2} \Big) \tag{A.4d}$$

$$\tilde{B}_{12} = 0\,, \qquad e^{2\tilde{\phi}} = e^{2\phi} H^{1/2}\,, \tag{A.4e}$$

where

$$D^{-1} = \cos^2\theta + \sin^2\theta H^{-1} \,. \tag{A.5}$$

Finally, we make a T-duality transformation in $\tilde{y}^2$ and we obtain

$$\begin{aligned}
\hat{g}_{11} &= \frac{1}{\tilde{g}_{22}}\Big(\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2\Big) - \frac{\tilde{B}_{12}^2}{\tilde{g}_{22}} = \frac{1}{\tilde{g}_{22}} = H^{-1/2}D \\
\hat{g}_{22} &= \frac{1}{\tilde{g}_{22}} = H^{-1/2}D\,, \qquad \hat{g}_{12} = -\frac{\tilde{B}_{12}}{\tilde{g}_{22}} = 0 \\
\hat{B}_{12} &= -\frac{\tilde{g}_{12}}{\tilde{g}_{22}} = \sin\theta\cos\theta\Big(H^{-1} - 1\Big)D = \tan\theta\Big(H^{-1}D - 1\Big) \\
e^{2\hat{\phi}} &= e^{2\tilde{\phi}}\frac{1}{\tilde{g}_{22}} = e^{2\phi}D
\end{aligned} \tag{A.6}$$

For the RR fields the T-duality transformations are in the most general case (with repeated application of the above prescription for disjoint pairs of spatial world-volume coordinates) a bit more involved [29], but as an illustration we work out the first application on coordinates y^1 and y^2. We start with the RR-field $A_{t12\cdots p}$ and T-duality in y^2 gives

$$A'_{t13\cdots p} = -A_{t13\cdots p2} \,. \tag{A.7}$$

Applying the rotation (A.3) one obtains

$$\tilde{A}_{t13\cdots p} = \cos\theta A'_{t13\cdots p} = -\cos\theta A_{t13\cdots p2} \tag{A.8}$$

$$\tilde{A}_{t23\cdots p} = -\sin\theta A'_{t13\cdots p} = \sin\theta A_{t13\cdots p2} \tag{A.9}$$

so that the final T-duality in $\tilde{y}^2$ gives

$$\begin{aligned}
\hat{A}_{t3\cdots p} &= -\tilde{A}_{t3\cdots p2} = -(-1)^p\tilde{A}_{t2\cdots p} = -(-1)^p\sin\theta A_{t13\cdots p2} = -\sin\theta A_{t12\cdots p} \\
\hat{A}_{t12\cdots p} &= (-1)^p\hat{A}_{t13\cdots p2} = (-1)^p\left[-\tilde{A}_{t13\cdots p} + \tilde{A}_{t23\cdots p}\frac{\tilde{g}_{12}}{\tilde{g}_{22}}\right] \\
&= \Big(\cos\theta + \sin\theta\frac{\tilde{g}_{12}}{\tilde{g}_{22}}\Big)A_{t12\cdots p} = \cos\theta D A_{t12\cdots p}\,,
\end{aligned} \tag{A.10}$$

where we used (A.4d), (A.4c) and the definition (A.5).

B. RR gauge potentials

In this appendix we give the RR potentials of the asymptotically-flat spinning bound state solution (2.2)–(2.4), which can be obtained from the procedure outlined in appendix A using in particular the T-duality transformations of the RR fields given in ref. [29]. To write these expressions we define the following one-form, relevant for spinning brane solutions

$$I = \frac{1}{\sinh\alpha}\Big(\cosh\alpha dt - \sum_{i=1}^{n} l_i \mu_i^2 d\phi_i\Big) \tag{B.1}$$

and we consider maximal rank $m = [p/2]$, since lower rank can be obtained by setting the appropriate θ_k to zero.

The RR fields for the cases $p = 2, \ldots, 6$ are then given by the following expressions: for the $p = 2$ (D2-D0) case we have

$$A_1 = -\sin\theta_1\Big(H^{-1} - 1\Big)I\,, \qquad A_3 = \cos\theta_1 D_1\Big(H^{-1} - 1\Big)I \wedge dy^1 \wedge dy^2 \tag{B.2}$$

and for $p = 3$ (D3-D1) analogously

$$A_2 = \sin\theta_1\Big(H^{-1} - 1\Big)I \wedge dy^3\,, \qquad A_4 = -\cos\theta_1 D_1\Big(H^{-1} - 1\Big)I \wedge dy^1 \wedge dy^2 \wedge dy^3\,. \tag{B.3}$$

The next case is $p = 4$ (D4-D2-D0) with RR-fields

$$\begin{aligned} A_1 &= \sin\theta_1 \sin\theta_2\Big(H^{-1} - 1\Big)I \\ A_3 &= -\Big(H^{-1} - 1\Big)I \wedge \Big[\cos\theta_1 D_1 \sin\theta_2 dy^1 \wedge dy^2 + \sin\theta_1 \cos\theta_2 D_2 dy^3 \wedge dy^4\Big] \\ A_5 &= \cos\theta_1 D_1 \cos\theta_2 D_2\Big(H^{-1} - 1\Big)I \wedge dy^1 \wedge \cdots \wedge dy^4 \end{aligned} \tag{B.4}$$

and analogously $p = 5$ (D5-D3-D1)

$$\begin{aligned} A_2 &= -\sin\theta_1 \sin\theta_2\Big(H^{-1} - 1\Big)I \wedge dy^5 \\ A_4 &= \Big(H^{-1} - 1\Big)I \wedge \Big[\cos\theta_1 D_1 \sin\theta_2 dy^1 \wedge dy^2 + \sin\theta_1 \cos\theta_2 D_2 dy^3 \wedge dy^4\Big] \wedge dy^5 \\ A_6 &= -\cos\theta_1 D_1 \cos\theta_2 D_2\Big(H^{-1} - 1\Big)I \wedge dy^1 \wedge \cdots \wedge dy^5\,. \end{aligned} \tag{B.5}$$

Finally, for $p = 6$ (D6-D4-D2-D0) the RR fields are

$$\begin{aligned} A_1 &= -\sin\theta_1 \sin\theta_2 \sin\theta_3\Big(H^{-1} - 1\Big)I \\ A_3 &= \Big(H^{-1} - 1\Big)I \wedge \Big[\cos\theta_1 D_1 \sin\theta_2 \sin\theta_3 dy^1 \wedge dy^2 + \\ &\quad + \sin\theta_1 \cos\theta_2 D_2 \sin\theta_3 dy^3 \wedge dy^4 + \sin\theta_1 \sin\theta_2 \cos\theta_3 D_3 dy^5 \wedge dy^6\Big] \\ A_5 &= -\Big(H^{-1} - 1\Big)I \wedge \Big[\sin\theta_1 \cos\theta_2 D_2 \cos\theta_3 D_3 dy^3 \wedge \cdots \wedge dy^6 + \\ &\quad + \cos\theta_1 D_1 \sin\theta_2 \cos\theta_3 D_3 dy^1 \wedge dy^2 \wedge dy^5 \wedge dy^6 + \\ &\quad + \cos\theta_1 D_1 \cos\theta_2 D_2 \sin\theta_3 dy^1 \wedge \cdots \wedge dy^4\Big] \\ A_7 &= \cos\theta_1 D_1 \cos\theta_2 D_2 \cos\theta_3 D_3\Big(H^{-1} - 1\Big)I \wedge dy^1 \wedge \cdots \wedge dy^6\,. \end{aligned} \tag{B.6}$$

Note that in the extremal and non-rotating case, the odd cases $p = 3, 5$ can be obtained from the even cases $p = 2, 4$ by a T-duality in y^3 and y^5 respectively.

For the case $p = 6$, in which the B-field can have the largest rank, we also list the charges and chemical potentials associated to the D6, D4, D2 and D0-branes respectively,

$$\mu_6 = \cos\theta_1 \cos\theta_2 \cos\theta_3 \mu\,, \qquad Q_6 = \cos\theta_1 \cos\theta_2 \cos\theta_3 Q \tag{B.7a}$$

$$\mu_4^{(\alpha)} = -\sin\theta_\alpha \cos\theta_\beta \cos\theta_\gamma \mu\,, \qquad Q_4^{(\alpha)} = -\sin\theta_\alpha \cos\theta_\beta \cos\theta_\gamma Q \tag{B.7b}$$

$$\alpha = 1, \ldots, 3$$

$$\mu_2^{(\alpha)} = \cos\theta_\alpha \sin\theta_\beta \sin\theta_\gamma \mu\,, \qquad Q_2^{(\alpha)} = \cos\theta_\alpha \sin\theta_\beta \sin\theta_\gamma Q \tag{B.7c}$$

$$\alpha = 1, \ldots, 3$$

$$\mu_0 = -\sin\theta_1 \sin\theta_2 \sin\theta_3 \mu\,, \qquad Q_0 = -\sin\theta_1 \sin\theta_2 \sin\theta_3 Q \tag{B.7d}$$

where for each α in (B.7b), (B.7c) $\beta < \gamma$ are both not equal to α, and μ and Q are the thermodynamic quantities

$$\mu = \tanh\alpha\,, \qquad Q = \frac{V_p V(S^{8-p})}{16\pi G} r_0^{7-p}(7-p)\sinh\alpha\cosh\alpha \tag{B.8}$$

with $p = 6$. For the other cases $p = 2, \ldots, 5$ the results follow the same pattern, and we have in general that

$$\sum_{k=0}^{m} \sum_{\alpha} \mu_{p-2k}^{(\alpha)} Q_{p-2k}^{(\alpha)} = \mu Q\,, \tag{B.9}$$

where α labels the different embeddings of the D$(p-2k)$-brane into the Dp-brane.

The near-horizon limit of these gauge potentials is most easily expressed in terms of the T-duality invariant combinations

$$\mathcal{A}_q = (e^{-B_2} A)_q = A_q - B_2 A_{q-2} + \frac{1}{2} B_2^2 A_{q-4} + \cdots \tag{B.10}$$

where it is important to keep the constant parts in the gauge potentials when present. The rescaled RR gauge potentials in the near-horizon limit (3.1), (3.2) are then

$$\mathcal{A}_{p+1-2k} = \frac{(\mathcal{A}_{p+1-2k})_{\text{old}}}{l_s^{4+2m-2k}}\,, \qquad k = 0, \ldots, m \tag{B.11}$$

and it can be checked that the new quantities are indeed finite[11] in the limit $l_s \to 0$.

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